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Estimate of Peirce's Linear Associative Algebra.

BY H. E. HAWKES.

In the fourth volume of this journal appeared a remarkable memoir by Benjamin Peirce* in which the author made the first systematic attempt to classify and enumerate hyper-complex number systems. Though his work attracted wide and favorable comment in England† and America at the time, continental investigators on the subject during the last fifteen years have given him scarcely the credit which his results and his methods deserve. Adverse criticism has been due in part to a misunderstanding of Peirce's definitions, in part to the fact that certain of Peirce's principles of classification are entirely arbitrary and quite distinct in statement from those used by Study and Scheffers, in part to Peirce's vague and in some cases unsatisfactory proofs, and finally to the extreme generality of the point of view from which his memoir sprang, namely a "philosophic study of the laws of algebraic operation." If Peirce's work is to receive its due recognition, three questions must be discussed. *First.* What problem did Peirce attack, and to what extent did he solve it? *Second.* What relation does this problem bear to that treated by Study and Scheffers? *Third.* To what extent do Peirce's methods assist in the solution of that problem?

In this paper I propose to discuss the first and second questions. The third question the writer considers elsewhere.‡

Historical Review.

With the discovery and general use of the geometrical interpretation of the ordinary complex number, the question naturally arose whether a number system

* "Linear Associative Algebra," American Journal of Mathematics, vol. IV, p. 97, 1881. This memoir was read before the National Academy of Sciences in 1870 and a few lithographed copies were made in the same year.

† Spottiswoode, Proceedings of the London Mathematical Society, vol. IV, p. 152, 1872. Cayley, Collected Works, XII, p. 465; XI, p. 457.

‡ Transactions of the American Mathematical Society, vol. III.

could be found which was susceptible of an interpretation in space, somewhat analogous to that of the ordinary complex numbers in the plane. Gauss* felt the need of and attempted to form such a system. Whether he did or did not anticipate Hamilton in the discovery of quaternions,† his work in that direction has not until recently been published‡ and could have had no influence on Peirce's investigations. That Gauss was very familiar with certain properties of hypercomplex numbers is evident from his well-known query regarding the inadmissibility into arithmetic of number systems in more than two units, which stimulated Hankel,§ Weierstrass,|| and many others to make investigations in this field.

In 1835 Hamilton¶ developed a complete theory of ordinary complex numbers in two units under the name of number couples. His definitions of complex numbers and the operations on them though ostensibly associated with "steps" of time have no immediate logical connection with the intuition** of time, and may be considered (essentially) as a discussion of pure complex numbers. De Morgan†† in his Double Algebra took the short step necessary to place complex numbers on a purely symbolic basis, in a form now very generally adopted‡‡. After defining equality and the operations of addition and subtraction of two complex numbers, Hamilton discusses§§ the operation of multiplication He defines the product

$$(0,\, b_1)(0,\, b_2) = (\gamma_{121} b_1 b_2 + \gamma_{122} b_1 b_2),$$

where γ_{121}, γ_{122} are arbitrary constants independent of b_1 and b_2. He proceeds to investigate the values that these "constants of multiplication" can have without introducing divisors of zero into the system, and determines on $\gamma_{121} = -1$, $\gamma_{122} = 0$ as the simplest possible values. This evidently leads him to the ordinary complex number system. This is, so far as I know, the first use of the

* Klein, Göttinger Nachrichten, 1898, p. 15; Tait, Proceedings of the Royal Society of Edinburgh, vol. XXIII.

† Life of Sir William Rowan Hamilton, by Graves, vol. III, p. 312.

‡ Werke, vol. VIII, p. 358.

§ "Theorie der Complexen Zahlensysteme," 1867, §29.

|| Göttinger Nachrichten, 1884, p. 395.

¶ Transactions of the Royal Irish Academy, vol. XVII, p. 293.

** De Morgan pointed out this fact in the Transactions of the Cambridge Philosophical Society, vol. VII, p. 175, 1842.

†† "Trigonometry and Double Algebra," London, 1849.

‡‡ Burkhardt, "Analytische Functionen," pp. 1–9; Study, Encyklopädie der mathematischen Wissenschaften, vol. I, p. 149; Weber, "Lehrbuch der Algebra," Einleitung, p. 19.

§§ Hamilton, loc. cit., p. 400.

constants of multiplication of a number system. In 1843* Hamilton determines the values of the sixty-four constants of multiplication in quaternions. Hamilton does not at this time consider systems, in three units, but it appears† that as early as 1834 he was attempting to find a system in three units which would have a useful geometric interpretation, but was deterred by the presence of divisors of zero which he correctly believed existed in every such system. The fact that quaternions contained no divisors of zero Hamilton considered one of its chief merits.‡

In 1843§ Hamilton first discusses number system in n units. He defines operations on such numbers and gives the condition

$$\sum_{s=1}^{n} \gamma_{iks}\,\gamma_{sjt} = \sum_{s=1}^{n} \gamma_{kjs}\,\gamma_{ist}, \qquad (i, k, j, t = 1 \,\ldots\, n) \quad (1)$$

that a system be associative.‖ In his treatment of division he does not arrive at divisors of zero explicitly, but contents himself with the remark that "the n sought coefficients of the symbolic quotient can *in general* be determined."¶ (The italics are mine.) Had he considered the case where the coefficients of the desired quotient could not be determined, he would have been lead to a discussion of divisors of zero. Hamilton attempted to develop the theory of, and perhaps to classify, systems in n units, but the existence of the n^4 equations (1) that the constants of multiplication of an associative system must satisfy seemed to offer insuperable difficulties.** Hamilton was considerably embarrassed in his early work†† by the lack of a useful definition of equality of two hypercomplex numbers. The assumption that

$$\sum_{1}^{n} a_i\, e_i = \sum_{1}^{n} b_i\, e_i$$

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* Transactions of the Royal Irish Academy, vol. XXI, p. 241.

† "Lectures on Quaternions," preface, p. 16, also p. 20.

‡ Life of Hamilton, vol. III, p. 254.

§ Transactions R. I. A., vol. XXI, p. 231.

‖ Ibid., p. 231; Schubert, (Encyc. math. Wiss., vol. I, p. 7) and Hankel (loc. cit., p. 3) state that the term "associative" goes back to Hamilton but give no reference. The word is used for the first time by Hamilton in Proceedings of the Royal Irish Academy, vol. II, p. 430.

¶ Transactions R. I. A., vol. XXI, p. 227.

** "Lectures on Quaternions," preface, p. 31.

†† Life of Hamilton, vol. III, p. 247.

when, and only when,

$$a_i = b_i, \qquad (i = 1, \ldots, n)$$

de Morgan* first gives for $n = 3$.

De Morgan, in 1844,† discusses the general commutative and associative systems with a modulus in three units. He arrives at several particular systems which he considers distinct though in somec ases they are equivalent, and gives several examples of divisors of zero. His systems, as well as those considered by Hamilton, have real parameters.

The pluquaternions of Kirkman,‡ the clefs of Cauchy,§ Grassmann's "Ausdehnungslehre," Hankel's|| alternating numbers, and the Situation Kalcul of Scheffler make little or no advance in the problem of classifying number systems, though Grassmann admits complex parameters. Important investigations were made by Frobenius, C. S. Peirce, and Clifford, subsequent to the appearance of Peirce's memoir, and on that account are not noticed here.

Peirce's Problem.

We have seen that previous to 1870 the number in n units and operations on it were defined, and the conditions imposed on the constants of multiplication that a closed system be associative were stated. There was, however, no method suggested by which one closed system could be grouped with an infinite number of other systems and a classification of all closed systems arrived at. The great usefulness of certain systems as Cauchy's clefs and quaternions would indicate that by an exhaustive search after all number systems, others might be found which would be adapted to various domains of research. The problem that Peirce set for himself was:**

1°. To enumerate exhaustively all types of number systems for a small number of units.

2°. To develop a calculus for some or all of these systems.

3°. To make application of these calculuses.

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* Transactions of the Cambridge Philosophical Society. vol. VII, p. 177, 1842.

† Trans. Cam. Philos. Soc., vol. VIII, p. 241, 1844.

‡ Philosophical Magazine, 3rd Series, vol. XXXIII, pp. 447 and 494, 1848.

|| Comptes Rendus, vol. XXXVI; pp. 70 and 129, 1853.

§ Loc. cit., p. 119.

** Loc. cit., pp. 98 and 119.

The first part of the problem occupies practically all of Peirce's published work on the subject.

In order to discuss the two questions proposed at the beginning of the paper, I shall treat in turn the principles of classification used by Peirce, and show their relation to those employed by more recent investigators. These principles of classification are five in number.

I. *Number* of *Units.*—Systems in the same number of units are thrown into the same group. This principle Peirce uses in common with all other writers on the subject.

II. *Equivalence.*—Two systems in the same number of units, $e_1, e_2 \ldots\ldots e,$ and $e'_1, e'_2 \ldots\ldots e'_n$, are considered equivalent if linear relations exist of the type

$$e'_k = \sum_{i=1}^{n} a_{ki} e_i, \qquad k = 1 \ldots\ldots n.$$

Peirce evidently assumes, though not explicitly, that $|a_{ki}| \neq 0$, and that for any value of k, the coefficient $a_{kk} \neq 0$. The latter unnecessary restriction is not always followed by the editor, Mr. C. S. Peirce.* This devise of classification Peirce justly considers one of the leading elements of originality in his work This principle has been used by most subsequent writers.†

III. *Pure Systems.*—Peirce further distinguishes between pure and mixed systems, but proposes to enumerate only the former. His definitions of these terms are not precise, but an inspection of their use shows that a system is called mixed if its units may be divided into two or more groups (which may have common units), such that the product, if non-vanishing, of two units of the same group is a number in that group, while the product of units which are not found in the same group is zero.‡ For illustration, on p. 123 [13], we have the system

	i	j	k
i	i	j	0
j	0	0	0
k	k	0	0

* Loc. cit., for example, p. 136 (foot-note); p. 205 (foot-note).

† Equivalent systems are of the same "*typus*" (see Study, Göttinger Nachrichten, 1889, p. 240).

‡ Cayley gives substantially this definition for the case $n = 2$, Collected Works, XII, p. 61.

In this mixed system the groups are i, j and i, k. Evidently the product inside each group is in that group while the products of j and k vanish. On p. 129 [12], we have

	i	j	k	l
i	i	j	k	l
j	j	k	0	0
k	k	0	0	0
l	0	0	0	0

In this mixed system the groups are i, j, k and i, l.

Again, on p. 133 [162] we have

	i	j	k	l
i	i	j	0	0
j	0	0	0	j
k	k	0	0	0
l	0	0	k	l

The groups here are i, j; i, k; l, k, and l, j.

Peirce's definitions of pure and mixed algebras correspond exactly to the definitions of irreducible and reducible number systems used by Scheffers,* except that the groups in a reducible system can contain no common units. Thus in Scheffer's enumeration appear several systems that Peirce rejects as not pure. In fact, every system given by Study or Scheffers which is not included in Peirce's enumeration he rejects as mixed, either in particular or under some general rule. An understanding of the definitions under which Peirce placed himself renders some of the criticisms of his work less forcible. For example, Study† states that Peirce gives only one of the five existant distinct systems in

*Mathematische Annalen, vol. 39, p. 317.

† Göttinger Nachrichten, 1889, p. 240.

three units. On comparing the systems given by the two writers, we see that all but one, (III), of Study's* systems are mixed and are rejected by Peirce on that ground. This system Peirce gives.

IV. *Reciprocity.*—Study† has defined two systems to be reciprocals of each other when the multiplication table of one is derived from that of the other by an interchange of rows and columns. Peirce gives no definition corresponding to this, but his usage follows it rather closely, and he assumes throughout his memoir that reciprocal systems are "virtual repetitions" of each other; for example, on p. 121, Peirce derives the system in two units

$$\begin{array}{c|c|c|} & i & j \\ \hline i & i & j \\ \hline j & 0 & 0 \\ \hline \end{array} \tag{1}$$

and states that

$$\begin{array}{c|c|c|} & i & j \\ \hline i & i & 0 \\ \hline j & j & 0 \\ \hline \end{array} \tag{2}$$

is a "virtual repetition." Notwithstanding the fact that Peirce assumes continually that such systems are not to be distinguished from each other, Cayley‡ complains that (2) is lacking since (1) and (2) are not equivalent. Bucheim§ in attempting to show what Peirce means by "virtual repetition" and to justify his position, arrives at a necessary and sufficient condition that two systems be reciprocals. In only one respect Peirce does fail to follow the present usage in regard to reciprocal systems. If a subsystem in one system is the reciprocal of a subsystem of another, while in other respects the systems are identical, Peirce calls them "virtual repetitions" of each other. For example, on p. 154 [111] the following systems are not considered distinct, since the subsystems formed by the units l, m in the systems are reciprocal.

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* Loc. cit., p. 246.

† Loc. cit., p. 240.

‡ Collected Works, XII, p. 61.

§ Messenger of Mathematics, vol. XV, 1886.

	i	j	k	l	m
i	i	j	0	0	0
j	0	0	$a_{23}i$	$b_{24}j$	$b_{25}j$
k	k	$d_{32}l+e_{32}m$	0	0	0
l	0	0	$c_{43}k$	l	m
m	0	0	$c_{53}k$	0	0

	i	j	k	l	m
i	i	j	0	0	0
j	0	0	$a_{23}i$	$b_{24}j$	$b_{25}j$
k	k	$d_{32}l+e_{32}m$	0	0	0
l	0	0	$c_{43}k$	l	0
m	0	0	$c_{53}k$	m	0

V. *Idempotent Numbers.*—Peirce defines an idempotent number as one, say α, for which a positive integer m exists so that $\alpha^m = \alpha$. If we define the modulus of a system as the number μ such that $x\mu = \mu x = x$ for every number x in the system, we see that the existence of an idempotent number is the necessary, though not sufficient, condition to the existence of a modulus. Peirce divides his systems into two groups, one of which contains systems with an idempotent, while the other contains systems without such a number. The former systems he considers of chief importance, and derives other systems only to assist in obtaining systems with an idempotent number. Thus he obtains all pure, inequivalent systems with a modulus, and several with no modulus: for example, each of the systems f_4, e_4, p. 130; h_4, p. 133, possesses an idempotent number, but no modulus.

The very close relation that exists between systems with idempotent units and those with a modulus, and the modifications necessary to make in Peirce's method, to derive all such systems are indicated elsewhere by the writer.*

We can now state precisely the problem that Peirce set for himself. He aimed to develop so much of the theory of hypercomplex numbers as would

* Loc. cit.

enable him to enumerate all inequivalent, pure, non reciprocal number systems in less than seven units. The relation to the problem treated by Scheffers is plain, if we remember that the first two of Peirce's principles of classification are identical with those of Scheffers, and the other three are only slightly modified. Peirce solved this problem completely. The theorems stated by him are in every case true, though in some cases his proofs are invalid.

It is remarkable that Peirce did not avail himself of the clear and compact definitions of equality, and the fundamental operations given by De Morgan and Hamilton. Had this course been followed it must have lead to a lucidity of development the lack of which Study and Molien justly deplore.

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